

A PROBLEM OF AMBIENCE

Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

WILLIAM KELSO MORRILL

JUNE, 1929

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A Problem of Ambience.

BY WILLIAM KELSO MORRILL.

In the following paper, we shall consider a triangle of directed lines, the vertices of which are moving with the same constant speed parallel to their respective opposite sides but in opposite directions. The invariants of the triangle are studied, and the motions of the vertices are investigated by the aid of the Weierstrass elliptic function theory as well as the q -series of Jacobi.

1. *The Invariants of the Triangle.* Let a, b, c be the lengths of the sides of the triangle, and α, β, γ their respective directions. If θ, ϕ, ψ are the angles which the sides of the triangle make with the base line, then $-\alpha = e^{i\theta}$; $\beta = e^{i\phi}$; $-\gamma = e^{i\psi}$. We shall use A, B, C in two senses: first as the affices of the vertices, second as the interior angles of the triangle. The motion of the vertices is given by the following differential equations

$$\dot{A} = -\alpha v, \quad \dot{B} = -\beta v, \quad \dot{C} = -\gamma v,$$

where the dot indicates differentiation with respect to the time and v is the speed. We can now write $a\alpha = C - B$. Then

$$D_t a\alpha \equiv \dot{a}\alpha + \dot{\alpha}a = \dot{C} - \dot{B} = (\beta - \gamma)v$$

$$\text{and} \quad \dot{a} + (\dot{\alpha}/\alpha)a = (\beta/\alpha - \gamma/\alpha)v.$$

$$\text{But} \quad \beta/\alpha = e^{i(\pi-C)} = -\cos C + i\sin C$$

$$\text{and} \quad \gamma/\alpha = e^{i(\pi+B)} = -\cos B - i\sin B.$$

$$\therefore \quad \dot{a} + a\dot{\alpha}/\alpha = v[\cos B - \cos C + i(\sin B + \sin C)].$$

Equating reals and imaginaries,

$$\begin{aligned} \dot{a} &= (\cos B - \cos C)v, \\ a\dot{\alpha}/\alpha &= i(\sin B + \sin C)v. \end{aligned}$$

The variations of the sides of the triangle are given, therefore, by

$$1.1 \quad \left\{ \begin{aligned} \dot{a} &= (\cos B - \cos C)v, \\ \dot{b} &= (\cos C - \cos A)v, \\ \dot{c} &= (\cos A - \cos B)v. \end{aligned} \right.$$

Now $v = d\Lambda/dt$, where Λ is the distance each body moves in the time t . Choosing $v = 1$, we have $d\Lambda = dt$ and $\Lambda = t$. Adding equations 1.1, we have $\dot{a} + \dot{b} + \dot{c} = 0$.

$$1.2 \quad \therefore \quad a + b + c = s_1$$

where s_1 is a constant; and our first result is that the perimeter of the triangle remains constant. Finding the perimeter constant suggests a study of the area.

$$\text{Area} = [s(s-a)(s-b)(s-c)]^{1/2},$$

where $s = (a + b + c)/2$. Thus, letting

$$\begin{aligned} X &\equiv 16(\text{Area})^2 = 2(a^2b^2 + b^2c^2 + a^2c^2) - a^4 - b^4 - c^4 \\ \dot{X} &= 4\{(b^2 + c^2 - a^2)\dot{a}a + (c^2 + a^2 - b^2)\dot{b}b + (a^2 + b^2 - c^2)\dot{c}c\} \\ &= 8abc(\dot{a} \cos A + \dot{b} \cos B + \dot{c} \cos C). \end{aligned}$$

Substituting the values of $\dot{a}$, $\dot{b}$, $\dot{c}$ from 1.1; we have: $\dot{X} = 0$ and

$$1.3 \quad X = S_3, \text{ where } S_3 \text{ is a constant; that is, the area also is constant.}$$

We thus find *the perimeter and area of the triangle are invariant under the motion.*

2. *Introducing the Elliptic Functions.* Let $x = s - a$, $y = s - b$, $z = s - c$. Then from 1.2 and 1.3 respectively, we obtain

$$\begin{aligned} x + y + z &= k_1, & \text{and} \\ xyz &= k_3. \\ \therefore \quad dx + dy + dz &= 0, & \text{and} \\ yzdx + xzdy + xydz &= 0. \end{aligned}$$

From these two equations, we obtain

$$\begin{aligned} 2.1 \quad dx/x(y-z) &= dy/y(z-x) = dz/z(x-y) = d\mu, \\ \therefore dz/d\mu &= z(x-y) = z\{(x+y)^2 - 4xy\}^{1/2} = z[(k_1 - z)^2 - 4k_3/z]^{1/2} \\ (dz/d\mu)^2 &= z^2[(k_1 - z)^2 - 4k_3/z]. \end{aligned}$$

Put $z = -1/v$; then $dz/d\mu = v^2 dv/d\mu$, and

$$(dv/d\mu)^2 = (k_1 v + 1)^2 + 4k_3 v^3.$$

Now putting $v = w - k_1^2/12k_3$, we get

$$\begin{aligned} 2.2 \quad (dw/d\mu)^2 &= 4k_3 w^3 + (2k_1 - k_1^4/12k_3)w \\ &\quad + (1 - 2k_1^3/12k_3 + 2k_1^6/3 \cdot 12^2 k_3^2). \end{aligned}$$

Finally putting $k_3^{1/2} d\mu = du$, and

$$2.3 \quad g_2 = (-2k_1/k_3)(1 - k_1^3/24k_3) \quad \text{and}$$

$$2.4 \quad g_3 = (-1/k_3)(k_1^6/216k_3^2 - k_1^3/6k_3 + 1) \quad \text{and}$$

substituting in 2.2 we obtain

$$2.5 \quad (dw/du)^2 = 4w^3 - g_2w - g_3.$$

Equation 2.5 is the elliptic relation $(p'u)^2 = 4p^3u - g_2pu - g_3$, and hence our problem is an elliptic function problem.

We may then write $w = p(u - \gamma)$, where γ is a constant. Since $z = -1/v$, and $v = w - k_1^2/12k_3$, $z = 1/[k_1^2/12k_3 - p(u - \gamma)]$. Setting

$$2.6 \quad k_1^2/12k_3 = p\tau$$

and noting, from 2.1, that x and y have expressions similar to z , we have:

$$2.7 \quad \begin{cases} x = 1/[p\tau - p(u - \alpha)], \\ y = 1/[p\tau - p(u - \beta)], \\ z = 1/[p\tau - p(u - \gamma)]. \end{cases}$$

Note that $z^{-1}dz/du = (x - y)/k_3^{1/2} = p'(u - \gamma)/[p\tau - p(u - \gamma)]^2$. This equals zero when $x = y$, which is at the half periods of the parallelogram of periods, since we know the function p' is zero there. Conversely when $u - \gamma$ is a half period, that is when

$$2.8 \quad u - \gamma \equiv m_1\omega_1 + m_2\omega_2, \quad \text{where}$$

$m_1, m_2 = 0, 1, 2$ but $m_1 = m_2 \neq 0$ or 2 , $p'(u - \gamma) = 0$, and $x = y$ or from 2.7, $p(u - \alpha) = p(u - \beta)$.

Consider then $u - \alpha \equiv \beta - u$, and, therefore, $2u \equiv \alpha + \beta$. Since from 2.8 we have $2u \equiv 2\gamma$, it follows that

$$2.9 \quad \begin{aligned} \alpha + \beta &\equiv 2\gamma, & \beta + \gamma &\equiv 2\alpha, & \gamma + \alpha &\equiv 2\beta. \\ \therefore \quad \alpha - \gamma &\equiv 2\gamma - 2\alpha & \text{or} & & 3\alpha &\equiv 3\beta \equiv 3\gamma. \end{aligned}$$

This result tells us that α, β, γ in 2.7 are constants which differ from each other by thirds of a period. By choosing $\gamma = 0$, it follows that $\alpha = \alpha$, and $\beta = 2\alpha$, and we can rewrite 2.7 in the following way:

$$2.7' \quad \begin{cases} x = 1/(p\tau - pu), \\ y = 1/[p\tau - p(u + \alpha)], \\ z = 1/[p\tau - p(u - \alpha)]. \end{cases}$$

x has poles at $u = \pm \tau$; y has poles at $u = \pm \tau - \alpha$; and z has poles at $u = \pm \tau + \alpha$. Since $x + y + z$ is a constant, the sum of the poles of x, y , and z respectively must be a period. Hence τ must be a third of a period.

The type of network can now be determined quite easily: g_2 and g_3 are both real and the discriminant

$$\Delta \equiv g_2^3 - 27g_3^2 = (k_1^3 - 27k_3)/k_3^3 > 0.$$

This follows from the theorem: *If n numbers $x_1, \dots, x_n$ are positive, the arithmetical mean must be equal to or greater than the geometrical mean.** Since g_2 and g_3 are real and $\Delta > 0$, our net work is rectangular.

We are interested in how the triangle behaves as the elliptic parameter u moves in a rectangular cell. But there are limitations on how u shall move in the cell.

There are eight thirds of a period in a cell. Of these, only four give distinct values to pu , due to the evenness of the p function. At the vertices of the cell pu is infinite. Along the boundaries it is real. As we move along the rectangle of half periods, pu decreases from $+\infty$ to $-\infty$ and is real. u must vary along such a path as will keep k_1 and k_3 real and positive. To find k_1 and k_3 in terms of elliptic functions, we proceed as follows

$$x = 1/(p\tau - pu) = 1/[p\tau - 1/u^2 - c_2u^2 \dots]$$

and this equals zero for $u = 0$.

Expanding $p(u + \tau)$ in a Taylor's series we obtain for y the following:

$$\begin{aligned} y &= 1/[p\tau - p(u + \tau)] = 1/[p\tau - (p\tau + up'\tau + u^2p''\tau/2! + \dots)] \\ &= 1/[-up'\tau(1 + up''\tau/2! p'\tau + u^2p'''\tau/3! p'\tau + \dots)] \\ &= (-1/up'\tau)[1 - up''\tau/2! p'\tau - u^2p'''\tau/3! p'\tau + \dots]. \end{aligned}$$

In a similar way we obtain

$$z = (1/up'\tau)[1 + up''\tau/2! p'\tau - u^2p'''\tau/3! p'\tau + \dots];$$

whence it follows

$$x + y + z = p''\tau/p'^2\tau + \text{a function of } u.$$

when $u = 0$, this function vanishes. Hence since $x + y + z = k_1$ is a constant, we have

$$k_1 = p''\tau/p'^2\tau.$$

From 2.6 we have $k_3 = k_1^2/12p\tau = p''^2\tau/12p'^4\tau \cdot p\tau$. Since τ is a third of a period, we have $12p\tau p'^2\tau = p''^2\tau$; for from the identity $2pu + p(2u) = p''^2u/4p'^2u$, we obtain on letting $u = \tau$,

$$2p\tau + p(2\tau) = p''^2\tau/4p'^2\tau.$$

But $p\tau = p(2\tau)$, and hence $12p\tau \cdot p'^2\tau = p''^2\tau$. Putting this back in the expression for k_3 above we obtain

* Todhunter's *Algebra*, Page 422.

$$k_3 = 1/p'^2\tau.$$

To keep k_1 and k_3 real and positive, $p'\tau$ must be real, and $p''\tau$ must be real and positive. Both of these conditions hold when τ lies on the real axis. Furthermore the path along which u moves on the cell must keep $2.7'$ positive and real. There is only one choice: u must move along the path which joins the mid-points of the vertical sides of the cell. Thus in our problem $u = \omega_2 + v$, where v is real and varies from 0 to $2\omega_1$.

3. *The Isosceles Cases.* Knowing the path along which u must move, we will next determine when the triangle becomes isosceles. Let us consider first $x = y$. Then $p(u + \tau) = pu$, hence $u + \tau = \pm u + 2m_1\omega_1 + 2m_2\omega_2$. The case of interest here is the one leading to $2u + \tau = 2m_1\omega_1 + 2m_2\omega_2$; but $u = \omega_2 + v$,

$$\therefore 2\omega_2 + 2v + \tau = 2m_1\omega_1 + 2m_2\omega_2;$$

$$\text{thus } 2v + \tau = 0, \quad \text{whence } v = -\tau/2 = 5\tau/2,$$

$$\text{or } = 3\tau, \quad \text{whence } v = \tau,$$

$$\text{or } = 6\tau, \quad \text{whence } v = 5\tau/2,$$

$$\text{or } = 9\tau, \quad \text{whence } v = 4\tau = \tau.$$

Hence the sides a and b of the triangle become equal for two values of u ; viz.,

$$u = 5\tau/2 + \omega_2 \quad \text{and} \quad u = \tau + \omega_2.$$

Next we will consider $x = z$. Then $p(u - \tau) = pu$ and going through a similar argument we find the sides a and c of our triangle are equal when

$$u = \omega_2 + \tau/2, \quad \text{and} \quad u = \omega_2 + 2\tau.$$

Finally we have $y = z$ when $p(u + \tau) = p(u - \tau)$. In this case we find that

$$u = \omega_2 \quad \text{and} \quad u = \omega_2 + 3\tau/2,$$

which tells us that our triangle was initially isosceles as $b = c$.

We can sum up the results of this section thus: Starting isosceles, the triangle becomes isosceles at every sixth of a period as u moves along the path $u = \omega_2 + v$, starting with $v = 0$.

4. *The Positional Equations.* We shall determine Λ as a function of the elliptic parameter u . From 1.1 we have

$$da/d\Lambda = \cos B - \cos C = (-2k_1/abc)(s-a)[(s-c)-(s-b)]$$

$$\therefore da/d\Lambda = (-2k_1k_3/abc)[p'up'\tau/(pu - p\tau)^2].$$

Now $s - a = 1/(p\tau - pu)$. Therefore $da/du = -p'u/(p\tau - pu)^2$, and $d\Lambda/du = abc/2k_1k_3^{1/2}$; whence

$$d\Lambda/du = -k_3^{1/2}/2k_1 + k_3^{1/2}k_1^2/8k_3 - (k_3^{1/2}/2)[pu + p(u + \tau) + p(u - \tau)].$$

Calling $K = -k_3^{1/2}/2k_1 + k_3^{1/2}k_1^2/8k_3$, we have

$$d\Lambda = Kdu - (k_3^{1/2}/2)[pu + p(u + \tau) + p(u - \tau)]du.$$

$$4.1 \quad \therefore \Lambda = Ku + (k_3^{1/2}/2)[\xi u + \xi(u + \tau) + \xi(u - \tau)] + c.$$

To determine the constant of integration, let $v = 0$, then $u = \omega_2$ and $c = K\omega_2 + 3k_3^{1/2}\eta_2/2$.

Thus for a particular position of u along its path, we can determine the distance the affices have moved.

As the affices move, the rates of change of the angles θ , ϕ , and ψ are given by a set of equations (see Section 1) which we will call positional equations:

$$4.2 \quad \left\{ \begin{array}{l} ad\theta/d\Lambda = \sin B + \sin C, \\ b d\phi/d\Lambda = \sin C + \sin A, \\ cd\psi/d\Lambda = \sin A + \sin B. \end{array} \right.$$

Expressing these in terms of elliptic functions, we have

$$\begin{aligned} d\theta/d\Lambda &= (\sin B + \sin C)/a \\ &= [2(k_1k_3)^{1/2}/abc] [(b + c)/a]. \end{aligned}$$

We have already found

$$d\Lambda/du = abc/2k_1k_3^{1/2}$$

Hence

$$\begin{aligned} d\theta/du &= (1/k_1^{1/2}) [(b + c)/a] \\ &= (1/k_1^{1/2}) \{1 + 2/[k_1(p\tau - pu) - 1]\}. \end{aligned}$$

If we make the substitution

$$4.3 \quad pv_0 = p\tau - 1/k_1,$$

where pv_0 is a constant, we get $d\theta = (1/k_1^{1/2}) [1 + 2/k_1(pv_0 - pu)] du$.*

Multiplying both sides by $p'v_0$ and integrating, we obtain

$$\theta p'v_0 = (1/k_1^{1/2}) [up'v_0 + \frac{2}{k_1} (\log \frac{\sigma(u + v_0)}{\sigma(u - v_0)} - 2u\xi v_0)] + C_1.$$

But, by putting for pv_0 its value given in 4.3, in the identity

$$p'v_0 = 4p^3v_0 - g_2pv_0 - g_3$$

we find that $p'v_0 = 2i/k_1^{3/2}$.

* Halphen, *Traité des Fonctions Elliptiques*, Vol. 1, p. 185.

The three angles of the triangle are then given by the following equations:

$$4.4 \quad \begin{cases} i\theta = iu/k_1^{1/2} + \log \frac{\sigma(u+v_0)}{\sigma(u-v_0)} - 2u\xi v_0 + C_1, \\ i\phi = i(u+\tau)/k_1^{1/2} + \log \frac{\sigma(u+\tau+v_0)}{\sigma(u+\tau-v_0)} - 2(u+\tau)\xi v_0 + C_2, \\ i\psi = i(u-\tau)/k_1^{1/2} + \log \frac{\sigma(u-\tau+v_0)}{\sigma(u-\tau-v_0)} - 2(u-\tau)\xi v_0 + C_3. \end{cases}$$

The equations 4.4 are important since they tell us the position of the triangle for a particular value of the parameter u .

5. *Introducing the q -series.* The representation of the elliptic functions by the q -series was invented by Jacobi* and is most important for practical problems. His invention made it possible to express doubly periodic functions in an infinite series, the terms of which are singly periodic functions. The problem we are interested in is the study of the paths of the vertices and of the center of gravity of the triangle. First, however, we will express the results already obtained as q -series.

Since we know the network of periods is rectangular, let us choose a rectangle standing upon a smaller side.

Put $2\omega_1 = \pi \dagger$ and then $2\omega_2 = ir\pi$ where $r > 1$. Hence we have $q = e^{i\pi\omega_2/\omega_1} = e^{-r\pi}$, and the larger we take r the smaller q becomes.

pu expressed as a q -series is $\ddagger$

$$pu = -(\eta_1/\omega_1) + (\pi/2\omega_1)^2 1/\sin^2(\pi u/2\omega_1) \\ - 2(\pi/\omega_1)^2 \sum_{n=1}^{\infty} [nq^{2n}/(1-q^{2n})] \cos nu(\pi/\omega_1).$$

For $2\omega_1 = \pi$, we have

$$pu = -2\eta_1/\pi + 1/\sin^2 u - 8 \sum [nq^{2n}/(1-q^{2n})] \cos 2nu;$$

and for $\tau = \pi/3$, we obtain

$$p\tau = -2\eta_1/\pi + 1/\sin^2 \pi/3 - 8 \sum [nq^{2n}/(1-q^{2n})] \cos 2n(\pi/3).$$

we are interested in the values for pu obtained for u moving along a line from ω_2 to $\omega_2 + 2\pi$ or, what is the same thing, for $u = \omega_2 + v$.

* Jacobi, *Fundamenta Nova*; Halphen, *Traité des Fonctions Elliptiques*, Vol. 1, p. 425.

$\dagger$ When we have put $2\omega_1 = \pi$, our unit is fixed and we are talking about a particular triangle. To generalize we merely multiply x , y , and z by μ (an arbitrary constant), and the discussion is the same.

$\ddagger$ Halphen, *Traité des Fonctions Elliptiques*, Vol. 1, p. 426.

$$p(\omega_2 + v) = -2\eta_1/\pi - 8 \sum [nq^n/(1 - q^{2n})] \cos 2nv.*$$

Expressed as q -series, we have

$$\begin{aligned} 5.1 \quad & \begin{cases} x = 1/(p\tau - pu) = (3/4)[1 - 6q \cos 2v + q^2(15 + 6 \cos 4v) + \dots] \\ y = 1/[p\tau - p(u + \tau)] = (3/4)[1 - 6q \cos 2(v + \pi/3) \\ \quad + q^2(15 + 6 \cos 4(v + \pi/3) + \dots] \\ z = 1/[p\tau - p(u - \tau)] = (3/4)[1 - 6q \cos 2(v - \pi/3) \\ \quad + q^2(15 + 6 \cos 4(v - \pi/3) + \dots] \end{cases} \\ 5.2 \quad & k_1 = x + y + z = (9/4)(1 + q^2 + \dots) \\ 5.3 \quad & k_3 = xyz = (27/64)(1 + 18q^2 + \dots) \\ 5.4 \quad & g_2 = 4/3 + 320(q^2 + \dots)^\dagger \\ 5.5 \quad & g_3 = 8/27 - (2^6 \cdot 7/3)(q^2 + \dots)^\dagger \\ 5.6 \quad & \Delta = 2^{12}q^2 + \dots^\dagger \end{aligned}$$

We are now prepared to explain our choice of a rectangular cell standing upon a smaller side. If $q = 0$ we see from 5.6 that the discriminant is zero. But this says $k_1^3 = 27k_3$ or that $a = b = c$ which is the equilateral case. Once equilateral the triangle stays equilateral, and the vertices move on a circle. It is easy to show that the triangle will never become equilateral unless it is that way initially. We shall consider the nearly equilateral case, hence we want q to be small. We can also express Λ , θ , ϕ , and ψ as q -series. We had $d\Lambda/dv = abc/2k_1k_3^{1/2}$; hence, $d\Lambda/dv = (2/3^{1/2})(1 + 57q^2/4 + \dots)$.

$$5.7 \quad \therefore \Lambda = (2/3^{1/2})(1 + 57q^2/4 + \dots)v + \Lambda_0,$$

where $\Lambda_0 = 0$. Also

$$\begin{aligned} & d\theta/dv = (1/k_1^{1/2})\{1 + 2/[k_1(p\tau - pu) - 1]\}, \\ & d\theta/dv = (2/3)[2 - 9q \cos 2v - 3q^2/2 + 45q^2 \cos 4v/2 + \dots] \\ 5.8 \quad & \therefore \theta = 4v/3 - 3q \sin 2v - q^2v + 15q^2 \sin 4v/4 + \dots + \theta_0. \end{aligned}$$

We can choose our triangle to make $\theta_0 = 0$. In our original conditions when $v = 0$, $u = \omega_2$ and our triangle is isosceles. Let us choose our base line to be initially parallel to the side a . We must determine the constants of integration for ϕ and ψ .

$$\begin{aligned} \phi - k &= (4/3)v - 3q \sin 2(v + \pi/3) \\ & \quad - q^2(v + \pi/3) + (15/4)q^2 \sin 4(v + \pi/3) + \dots, \\ \phi_0 - k &= -3q \sin (2\pi/3) - q^2 \cdot \pi/3 + (15/4)q^2 \sin (4\pi/3) + \dots, \end{aligned}$$

whence

$$\begin{aligned} 5.9 \quad \phi - \phi_0 &= (4/3)v + 3q[\sin 2\pi/3 - \sin 2(v + \pi/3)] \\ & \quad - q^2v + (15/4)[\sin 4(v + \pi/3) - \sin (4\pi/3)] + \dots \end{aligned}$$

* Halphen, *Traité des Fonctions Elliptiques*, Vol. 1, p. 426.

† Harkness and Morley, *Theory of Functions*, pp. 322-324.

In the same way

$$5.10 \quad \psi - \psi_0 = (4/3)v - 3q[\sin 2\pi/3 + \sin 2(v - \pi/3)] \\ - q^2v + (15/4)[\sin 4(v - \pi/3) + \sin (4\pi/3)] + \dots$$

In order to determine ϕ_0 and ψ_0 consider

$$a = k_1 - 1/(p\tau - pu) \\ = 3/2 - (9/2)q \cos 2v + [45/2 - (9/2) \cos 4v] q^2 + \dots$$

For $v = 0$; $a = a_0$,

$$\therefore a_0 = (3/2)[1 - 3q + 12q^2 + \dots],$$

and in a similar way, we find

$$b_0 = c_0 = (3/2)[1 + 3q/2 + 33q^2/2 + \dots].$$

Hence

$$5.11 \quad \cos \phi_0 = \frac{a_0}{2b_0} = - \frac{1 - 3q + 12q^2 + \dots}{2 + 3q + 33q^2 + \dots}$$

From this we get

$$5.12 \quad e^{i\phi_0} + e^{-i\phi_0} = -1 + 9q/2 - 9q^2/4 + \dots$$

$$\therefore e^{i\phi_0} = \omega + (i3^{3/2}/2)\omega^2q + \dots,$$

$$\text{and} \quad i\phi_0 = \log (\omega + (i3^{3/2}/2)\omega^2q + \dots)$$

To evaluate ψ_0 , we note that it is equal to $-\phi_0$, and hence $\cos \psi_0 = \cos \phi_0$. When we solve 5.12 for $e^{i\phi_0}$, we have two roots resulting from a quadratic equation. They represent the values of $e^{i\phi_0}$ and $e^{i\psi_0}$ respectively. Hence

$$e^{i\psi_0} = \omega^2 - (i3^{3/2}/2)\omega q + \dots,$$

$$\text{and} \quad i\psi_0 = \log [\omega^2 - (i3^{3/2}/2)\omega q + \dots].$$

Formula 5.11 is very important in that it fixes the value of q once the initial lengths of the sides of the triangle are given.

6. *The Paths of the Vertices and Centroid.* First consider the path of A . $e^{i\theta}$ is the turn from the base line to the side a . A is moving along some path with a direction $e^{(\pi+\theta)i} = -e^{i\theta}$. Since this equals $dA/d\Lambda$, we have

$$dA/d\Lambda = \exp\{i[4v/3 - 3q \sin 2v - q^2v + 15q^2 \sin 4v/2 + \dots]\} \\ = -\exp(4iv/3)\{1 - 3iq \sin 2v \\ + q^2[-(9/4) - iv + (9/4) \cos 4v + (15i/4) \sin 4v] + \dots\}, \quad \text{and}$$

$$d\Lambda/dv = 2(1 + 57q^2/4 + \dots)/3^{1/2}.$$

$$\begin{aligned}\therefore dA/dv &= -(2/3^{3/2}) \exp [i(4/3)v] [1 - 3iq \sin 2v \\ &\quad + q^2(12 - iv + 9 \cos 4v/4 + 15i \sin 4v/4) + \dots] \\ &= (-2/3^{3/2}) \exp [i(4/3)v] \{1 - 3q[\exp(2iv) - \exp(-2iv)]/2 \\ &\quad + q^2[48 - 4iv + 12 \exp(4iv) - 3 \exp(-4iv)]/4 + \dots\}.\end{aligned}$$

Put $\exp(2iv) = t^3$, then $dv = 3dt/2it$, and

$$dA/dt = i3^{3/2}[t - 3q(t^4 - t^2)/2 + (q^2/4)(48t - 6t \log t + 12t^7 - 3t^5) + \dots].$$

$$\begin{aligned}6.1 \quad \therefore A &= A_0 + i3^{3/2}\{t^2/2 - (3q/10t)(t^6 + 5) \\ &\quad + (q^2/16)[102t^2 - 12t^2 \log t + 6t^8 + 3t^4] + \dots\}.\end{aligned}$$

We can determine A_0 by taking $A = 0$ when $t = 1$. This represents the path along which A moves. The logarithmic term tells us the path is not closed but continually shifts over the plane.

By a similar method the equations of the paths of B and C , expanded as far as the first degree term in q , are found to be

$$6.2 \quad B = B_0 + i3^{3/2} \left(\frac{\omega t^2}{2} - \frac{i3^{3/2}qt^2}{4} - \frac{3\omega q(\omega t^6 + 5\omega^2)}{10t} + \dots \right),$$

$$6.3 \quad C = C_0 + i3^{3/2} \left(\frac{\omega^2 t^2}{2} + \frac{i3^{3/2}qt^2}{4} - \frac{3\omega^2 q(\omega^2 t^6 + 5\omega)}{10t} + \dots \right)$$

where B_0 and C_0 are determined in the same manner as A_0 .

To obtain the path of the centroid g we have, noting that $3g = A + B + C$,

$$\text{that} \quad g = (A_0 + B_0 + C_0)/3 + 3^{3/2}q/2it + \dots,$$

$$\text{where} \quad A_0 + B_0 + C_0 = 3^{3/2}iq/2 + \dots$$

$$\text{Hence} \quad g = (3^{3/2}q/2i)[1/(t-1)] + \dots$$

VITA.

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